Construction of dynamical quadratic algebras

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Abstract

We propose a dynamical extension of the quantum quadratic exchange algebras introduced by Freidel and Maillet. It admits two distinct fusion structures. A simple example is provided by the scalar Ruijsenaars-Schneider model.

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1 Introduction

The notion of dynamical quantum algebra was introduced by Felder [1] and Gervais-Neveu [2] and subsequently studied and developed in [3, 4, 6]. The classical limit (dynamical r-matrices) first appeared in [7] and was later investigated in [8]; examples related to the Ruijsenaars-Schneider (RS) model [9] were particularly studied [10, 11]. In general, they are characterized by the existence, in the quantum (and classical) R-matrices, of supplementary parameters identified as coordinates on the dual of some particular Lie algebra $\mathfrak{h}$. These parameters occur as dynamical variables in the classical case, hence the name.

These investigations until now concentrated on particular dynamical extensions of the quantum group structure characterised by its quadratic exchange relation:

$$R_{12}\ T_1\ T_2 = T_2\ T_1\ R_{12}$$

They were recently understood as Drinfel'd twists of the quantum group [12]. The quantum R-matrix obeys a dynamical cubic equation (Gervais-Neveu-Felder (GNF) equation), generalizing [1, 2] the quantum Yang-Baxter (YB) equation [13, 14, 15, 16, 17]

We will describe here an extension to quadratic exchange algebras of the notion of quantum dynamical algebra.

These algebras are characterised by so-called braided exchange relations [20]:

$$A_{12}\ T_1\ B_{12}\ T_2 = T_2\ C_{12}\ T_1\ D_{12} \tag{1}$$

where the generators of the algebra sit in the entries of T viewed as a matrix in $\mathrm{End}(V)$ for a given "auxiliary" vector space V; A, B, C, D are c-number structure matrices acting on $V \otimes V$. V may have the structure of a loop space $V \otimes C(\lambda)$ in which case the structure matrices depend on complex spectral parameters λ_1, λ_2. As usual in this context, the indices $1, 2$ label the auxiliary vector spaces that the matrices act on. Many examples are known in the case where A, B, C, D depend only on spectral parameters, see [18]. Recently, a universal structure was proposed for the specific case of reflection algebras $A = C = B^\pi = D^\pi$ [19]. Using associativity to compare both ways of exchanging $T_1 T_2 T_3$ into $T_3 T_2 T_1$ leads naturally to YB-type equations on the A, B, C, D matrices as sufficient consistency conditions. In [20] Maillet

and Freidel wrote down 8 YB-equations which provide a sufficient set of 3-space exchange conditions for the quantum algebra (1). This case is hereafter refered to as " nondynamical".

The question now arises whether there exists a consistent way of dynamizing these 8 YB-equations (in the sense of Gervais-Neveu-Felder [1, 2]) and whether such dynamized quantum YB-equations can be interpreted as sufficient 3-space exchange conditions for a dynamical quadratic quantum algebra.

For simplicity we choose $\mathfrak{gl}(n)$ as underlying Lie algebra and its Cartan subalgebra as the Lie algebra supporting the dynamical parameters. We will here define a dynamical quadratic quantum algebra (DQQA) for a particular choice of zero-weight conditions of the R-matrix set A, B, C, D under the action of the Cartan subalgebra $\mathfrak{h}$. This choice is consistent – as we will see – with the specific structure of the classical and quantum R-matrices for RS-models [11] and provides the general algebraic frame for the construction of quantum RS-models proposed by Arutyunov-Chekov-Frolov (ACF) [5]. We will prove that such sufficient conditions for 3-space exchange of these DQQA's realize a dynamical version of the quadratic YB-equations derived in [20]. We will also describe the ACF example of DQQA coming from the scalar RS model. We will then describe two distinct coproduct-type structures for this DQQA, generalizing the coproduct structures described in [20]; these coproducts allow for building other spin-chain type models from the scalar one. We will finally define a classical limit of the DQQA and show that the scalar Ruijsenaars-Schneider classical r-matrix structure does realize this classical limit (see [5]).

2 Dynamical quadratic quantum algebras

We start by expliciting the "dynamical" notation. Let $\mathfrak{g}$ be a simple Lie algebra and $\mathfrak{h}$ a commutative subalgebra of $\mathfrak{g}$ of dimension n. (For an extension to noncommutative $\mathfrak{h}$ see [21].) Let us choose a basis $\{h^i\}_{i=1}^n$ of $\mathfrak{h}^*$ and let $\lambda = \sum_i \lambda_i h^i$ be an element of $\mathfrak{h}^*$. The dual basis is denoted in $\mathfrak{h}$ by $\{h_i\}_{i=1}^n$. For any differentiable function $f(\lambda) = f(\{\lambda_i\})$ one defines:

$$f(\lambda + \gamma h) = e^{\gamma \mathcal{D}} f(\lambda) e^{-\gamma \mathcal{D}}, \tag{2}$$

where

$$\mathcal{D} = \sum_i h_i \partial_{\lambda_i} \tag{3}$$

It can be seen that this definition yields formally

$$f(\lambda + \gamma h) = f(\{\lambda_i + \gamma h_i\}) = \sum_{m \geq 0} \frac{\gamma^m}{m!} \sum_{i_1, \ldots, i_m = 1}^{n} \frac{\partial^m f(\lambda)}{\partial \lambda_{i_1} \ldots \partial \lambda_{i_m}} h_{i_1} \ldots h_{i_m} \tag{4}$$

which is a function on $\mathbb{C}^n$ taking values in $\mathsf{U}(\mathfrak{h})$.

Armed with these definitions we propose the following dynamization of the algebra relations (1)

$$A_{12}(\lambda) T_1(\lambda) B_{12}(\lambda) T_2(\lambda + \gamma h_1) = T_2(\lambda) C_{12}(\lambda) T_1(\lambda + \gamma h_2) D_{12}(\lambda) \tag{5}$$

We require additional assumptions on the R-matrix

$$[h \otimes \mathbf{1}, B_{12}] = 0, \quad [\mathbf{1} \otimes h, C_{12}] = 0, \quad [h \otimes \mathbf{1} + \mathbf{1} \otimes h, D_{12}] = 0 \quad (\forall h \in \mathfrak{h}) \tag{6}$$

$$B_{12} = C_{21}, \quad A_{21} = A_{12}^{-1}, \quad D_{21} = D_{12}^{-1} \tag{7}$$

Zero-weight conditions (6) will be presently seen to be consistent with the dynamical shifts in (5). We expect that different consistent choices of "zero-weight conditions" (6) will exist, leading to different DQQA's, but we will not discuss it at this time. Assumptions (7) are general self-consistency conditions for form-invariance of (5) under exchange of labels 1 and 2.

Now the (sufficient) consistency conditions can be derived thanks to a change of point of view advocated in [20]. Instead of looking at T as a matrix multiplied by, say, A from left and B from right one can think of T as a bivector which is *acted upon* by A and B. To put it another way: the triple matrix product $A \cdot T \cdot B$ can be viewed either as a sum $\sum_{p,q} A_p^i T_q^p B_j^q$ – where T_q^p is viewed as a matrix element –, or as a sum $\sum_{p,q} A_p^i \left(B^t\right)_q^j T^{p,q}$ – where $T^{p,q}$ is seen as a coordinate of a bivector of which each factor is multiplied separately. Then we rewrite equation (5) as

$$A_{12} T_{\mathbf{11'}}^{t_{1'}} B_{1'2} T_{\mathbf{22'}}^{t_{2'}} (\lambda + \gamma h_{1'}) = T_{\mathbf{22'}}^{t_{2'}} C_{12'} T_{\mathbf{11'}}^{t_{1'}} (\lambda + \gamma h_{2'}) D_{1'2'} \tag{8}$$

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where bivector labels are indicated in bold. For the sake of simplicity, explicit dependence on dynamical parameters is omitted wherever possible. Using the commutation relations (6) (in fact here only the first two are needed) and the transposition on spaces $1'$ and $2'$ equation (8) can be recast into the form

$$\mathcal{R}_{11',22'}T_{\mathbf{11'}}T_{\mathbf{22'}}(\lambda + \gamma h_{1'}) = T_{\mathbf{22'}}T_{\mathbf{11'}}(\lambda + \gamma h_{2'}) \tag{9}$$

where $\mathcal{R}_{11',22'}$ is defined as

$$\mathcal{R}_{11',22'} = (C_{12'}^{t_{2'}})^{-1}(D_{1'2'}^{t_{1'}t_{2'}})^{-1}A_{12}B_{1'2}^{t_{1'}} \tag{10}$$

The compatibility condition for the algebra generated by the elements of T is derived as usual. Starting from

$$T_{\mathbf{33'}}T_{\mathbf{22'}}(\lambda + \gamma h_{3'})T_{\mathbf{11'}}(\lambda + \gamma h_{2'} + \gamma h_{3'})$$

one compares both ways (consistent by associativity) of obtaining

$$T_{\mathbf{11'}}T_{\mathbf{22'}}(\lambda + \gamma h_{1'})T_{\mathbf{33'}}(\lambda + \gamma h_{1'} + \gamma h_{2'})$$

by permutation of the T-s using the exchange relation (9).

Lemma 1 *A sufficient condition for the consistency of 3-space exchange of T-matrices with*

$$\mathcal{R}_{11',22'}T_{\mathbf{11'}}T_{\mathbf{22'}}(\lambda + \gamma h_{1'}) = T_{\mathbf{22'}}T_{\mathbf{11'}}(\lambda + \gamma h_{2'}) \tag{11}$$

is the following dynamical Yang-Baxter equation for $\mathcal{R}$:

$$
\begin{aligned}
&\mathcal{R}_{11',22'}(\lambda + \gamma h_{3'})\mathcal{R}_{11',33'}(\lambda)\mathcal{R}_{22',33'}(\lambda + \gamma h_{1'})\\
=\ &\mathcal{R}_{22',33'}(\lambda)\mathcal{R}_{11',33'}(\lambda + \gamma h_{2'})\mathcal{R}_{11',22'}(\lambda)
\end{aligned}
\tag{12}
$$

Our goal is now to deduce from (12) a set of consistent dynamical equations for the four components of the matrix $\mathcal{R}$. We will illustrate this by explicitly describing the first step of the process in Appendix A.

In the end we find that under assumptions (6) and (7) the nondynamical YB-equations obtained in [20] can be consistently dynamized as follows and this dynamization in turn assures that (12) is satisfied.

4

$$
\begin{array}{rcl}
A_{12}\ A_{13}\ A_{23} & = & A_{23}\ A_{13}\ A_{12} \qquad\qquad\ \ (13) \\
D_{12}(\lambda + \gamma h_3)\ D_{13}\ D_{23}(\lambda + \gamma h_1) & = & D_{23}\ D_{13}(\lambda + \gamma h_2)\ D_{12} \quad\ \ (14) \\
D_{12}\ B_{13}\ B_{23}(\lambda + \gamma h_1) & = & B_{23}\ B_{13}(\lambda + \gamma h_2)\ D_{12} \quad\ \ (15) \\
A_{12}\ C_{13}\ C_{23} & = & C_{23}\ C_{13}\ A_{12}(\lambda + \gamma h_3) \quad (16)
\end{array}
$$

It can be checked that (13) and (14) are precisely the consistency conditions for the BC algebras (15) and (16). For example starting with

$$
B_{14}\ B_{24}(\lambda + \gamma h_1)\ B_{34}(\lambda + \gamma h_1 + \gamma h_2)
$$

and using the exchange relation (15) one obtains

$$
B_{34}\ B_{24}(\lambda + \gamma h_3)\ B_{14}(\lambda + \gamma h_2 + \gamma h_3)
$$

in two different ways. These two ways yield the same result whenever (14) is satisfied. Note that (14) together with the zero-weight condition (6) is the usual GNF equation. By contrast, A obeys a non-dynamical Yang-Baxter equation a lthough it also contains the dynamical variables.

To summarize we now state

Theorem 1 *The exchange relations*

$$
A_{12}(\lambda)T_1(\lambda)B_{12}(\lambda)T_2(\lambda + \gamma h_1) = T_2(\lambda)C_{12}(\lambda)T_1(\lambda + \gamma h_2)D_{12}(\lambda)
$$

where

$$
[h \otimes \mathbf{1}, B_{12}] = 0, \quad [\mathbf{1} \otimes h, C_{12}] = 0, \quad [h \otimes \mathbf{1} + \mathbf{1} \otimes h, D_{12}] = 0 \quad (\forall h \in \mathfrak{h})(17)
$$

$$
B_{12} = C_{21}, \quad A_{21} = A_{12}^{-1}, \quad D_{21} = D_{12}^{-1} \qquad\qquad (18)
$$

together with the relations

$$
\begin{array}{rcl}
A_{12}\ A_{13}\ A_{23} & = & A_{23}\ A_{13}\ A_{12} \qquad\qquad\ \ (19) \\
D_{12}(\lambda + \gamma h_3)\ D_{13}\ D_{23}(\lambda + \gamma h_1) & = & D_{23}\ D_{13}(\lambda + \gamma h_2)\ D_{12} \quad\ \ (20) \\
D_{12}\ B_{13}\ B_{23}(\lambda + \gamma h_1) & = & B_{23}\ B_{13}(\lambda + \gamma h_2)\ D_{12} \quad\ \ (21) \\
A_{12}\ C_{13}\ C_{23} & = & C_{23}\ C_{13}\ A_{12}(\lambda + \gamma h_3) \quad (22)
\end{array}
$$

yield an associative dynamical quadratic algebra.

We now formulate two fusion structures on the quantum space.

Theorem 2 *Let T_{1q} be a representation of the algebra (5) on some Hilbert space H_q. Let $L_{1q'}, R_{1q'}$ denote a representation on another Hilbert space $H_{q'}$ of the following set of exchange relations :*

$$
\begin{aligned}
A_{12}\ L_1 L_2 &= L_2 L_1\ A_{12} & (23)\\
R_1\ B_{12}\ L_2(\lambda + \gamma h_1) &= L_2\ B_{12}\ R_1 \\
D_{12}\ R_1 R_2(\lambda + \gamma h_1) &= R_2 R_1(\lambda + \gamma h_2)\ D_{12}
\end{aligned}
$$

then

$$
T_{1,qq'} = L_{1q'} T_{1q} R_{1q'} \qquad (24)
$$

is a representation on $H_q \otimes H_{q'}$ of the algebra (5).

Similarly, if $L_{1q'}, R_{1q'}$ is a representation on a Hilbert space $H_{q'}$ of the exchange relations

$$
\begin{aligned}
A_{12}\ L_1 L_2 &= L_2 L_1\ A_{12}(\lambda + \gamma h) & (25)\\
R_1\ B_{12}\ L_2(\lambda + \gamma h_1) &= L_2\ B_{12}(\lambda + \gamma h)\ R_1 \\
D_{12}\ (\lambda + \gamma h) R_1 R_2(\lambda + \gamma h_1) &= R_2 R_1(\lambda + \gamma h_2)\ D_{12}.
\end{aligned}
$$

Then

$$
T_{1,qq'} = L_{1q'} T_{1q}(\lambda + \gamma h_{q'}) R_{1q'} \qquad (26)
$$

yields a representation of the same algebra (5) on the space $H_q \otimes H_{q'}$.

It is assumed in (25) that the algebra of which $L_{1q'}$ and $R_{1q'}$ are representations on $H_{q'}$ has an $\mathfrak{h}$-module structure, thereby making sense of the unindexed dynamical shift.

Proof: direct check of (5) by using the set of relations (23) or (25).

Strictly speaking we have here defined fusion procedures of represented T-matrices. In the sequel we will refer to these fusion structures simply as "coproducts" even though we cannot prove yet that there is a universal bialgebra structure behind them.

A straightforward representation of the first LR-exchange algebra (23) is provided on $H_{q'} = V$ by taking $L_{1q'} \equiv A_{12}$ and $R_{1q'} \equiv B_{12}$ or $L_{1q'} \equiv (A_{12}^{-1})^{t_2}$

and $R_{1q'} \equiv (B_{12}^{t_2})^{-1}$. The second LR algebra, too, has a simple representation in terms of the structure matrices. Namely, one can take $L_{1q'} \equiv C_{12}$ and $R_{1q'} \equiv D_{12}$ or $L_{1q'} \equiv (C_{12}^{t_2})^{-1}$ and $R_{1q'} \equiv (D_{12}^{-1})^{t_2}$. These representations by structure matrices are made possible by the fact that A and B (respectively C and D) obey three out of the four consistency requirements: (19), (21) and (22) realizing (23) (respectively (20), (21) and (22) realizing (25)). Let us remark here that both LR-algebras (23) and (25), therefore both coproducts, are identical in the nondynamical limit $\gamma \to 0$ to the single T^+, T^- algebra and its coproduct described in [20]. This can be understood if one notices that in the nondynamical case the consistency relations (13) - (16) admit a particular symmetry: $A_{i3} \leftrightarrow C_{i3}$ and $D_{i3} \leftrightarrow B_{i3}$ for $i = 1, 2$. This is no longer true in the case $\gamma \neq 0$.

2.1   An example

A concrete realization of the algebra (5) is given [5] by the elliptic RS model [9]. For the sake of simplicity we only consider here its rational limit. Let us define the structure matrices as:

$$A(\lambda) = 1 + \sum_{i \neq j} \frac{\gamma}{\lambda_{ij}} (E_{ii} - E_{ij}) \otimes (E_{jj} - E_{ji}) \tag{27}$$

$$B(\lambda) = C(\lambda)^\pi = 1 + \sum_{i \neq j} \frac{\gamma}{\lambda_{ij} - \gamma} E_{jj} \otimes (E_{ii} - E_{ij}) \tag{28}$$

$$D(\lambda) = 1 - \sum_{i \neq j} \frac{\gamma}{\lambda_{ij}} E_{ii} \otimes E_{jj} + \sum_{i \neq j} \frac{\gamma}{\lambda_{ij}} E_{ij} \otimes E_{ji} \tag{29}$$

where E_{ij} is the elementary matrix whose entries are $(E_{ij})_{kl} = \delta_{ik}\delta_{jl}$ and $\lambda_{ij} = \lambda_i - \lambda_j$. These matrices verify the consistency conditions (13)-(16). A scalar representation of the exchange algebra defined with these structure matrices is then provided by:

$$T(\lambda) = \sum_{ij} \frac{\prod_{a \neq i} (\lambda_{aj} + \tilde{\gamma})}{\prod_{a \neq j} \lambda_{aj}} E_{ij} \otimes \mathbf{1} \tag{30}$$

Taking this as a starting point, one can now use the coproducts described above to construct other, higher dimensional nonabelian representations of the algebra defined by (27) - (29) which should provide us with a suitable algebraic framework to define and study spin generalizations of the RS-model.

Theorem 2 indeed provides us with the dynamical version of the construction of a monodromy matrix for a spin chain model by successive products of R-matrices, using the coproduct structure of the quantum group.

3   The classical limit

For classical integrable systems the starting point is the following quadratic Poisson-bracket algebra [20, 22]

$$\{l_1, l_2\} = a_{12}l_1l_2 + l_1b_{12}l_2 - l_2c_{12}l_1 - l_1l_2d_{12} \tag{31}$$

where the Lax-matrix l is a function on the phase space taking values in $\mathrm{End}(V)$, V being a finite dimensional vector space. The matrices a, b, c, d that define the quadratic algebra are elements of $\mathrm{End}(V \otimes V)$. We say that the algebra is dynamic if these matrices actually depend on the phase space variables.

In order to ensure the antisymmetry of the Poisson-bracket we impose the following constraints on the structure matrices:

$$a + a^\pi = \alpha\mathcal{C}, \qquad d + d^\pi = \alpha\mathcal{C}, \qquad b^\pi = c \qquad (\alpha \in \mathbb{C}) \tag{32}$$

where, as usual, π denotes the permutation in $\mathrm{End}(V \otimes V)$, and $\mathcal{C}$ is the Casimir-operator, i.e. for the $\mathfrak{gl}_n$ case $\mathcal{C} = \sum_{i,j} E_{ij} \otimes E_{ji}$. In other words, we are allowed to modify a and d by adding the same multiple of $\mathcal{C}$ to both of them. The Poisson-bracket will not change, since $[\mathcal{C}, l_1l_2] = 0$. These conditions on a and d are slightly more relaxed than usual: the reason for this will become clear when we consider the RS model.

A well-behaved Poisson-bracket should also verify the Jacobi identity. This is equivalent to demanding that the following general identity holds:

$$
\begin{aligned}
([a_{12}, a_{13}] + [a_{12}, a_{23}] + [a_{13}, a_{23}])\, l_1l_2l_3 - \\
-l_1l_2l_3\left([d_{12}, d_{13}] + [d_{12}, d_{23}] + [d_{13}, d_{23}]\right) + \\
+l_1l_2\left([d_{12}, b_{13}] + [d_{12}, d_{23}] + [b_{13}, b_{23}]\right) l_3 + \mathrm{circ.\ perm.} \\
-l_3\left([a_{12}, c_{13}] + [a_{12}, c_{23}] + [c_{13}, c_{23}]\right) l_1l_2 - \mathrm{circ.\ perm.} \\
-l_1l_2\{d_{12}, l_3\} - l_2l_3\{d_{23}, l_1\} - l_3l_1\{d_{31}, l_2\} \\
+\{a_{12}, l_3\}l_1l_2 + \{a_{23}, l_1\}l_2l_3 + \{a_{31}, l_2\}l_1l_3 \\
-l_2\{c_{12}, l_3\}l_1 - l_3\{c_{23}, l_1\}l_2 - l_1\{c_{31}, l_2\}l_3 \\
+l_1\{b_{12}, l_3\}l_2 + l_2\{b_{23}, l_1\}l_3 + l_3\{b_{31}, l_2\}l_1 = 0 \tag{33}
\end{aligned}
$$

where the last four lines appear because of the dynamical nature of the structure matrices.

3.1 An example: the hyperbolic Ruijsenaars-Schneider model

Due to the appearance of dynamical terms of the generic form $\{a, L\}$ in the Jacobi identity, it is not clear how to characterize general algebraic structures in (33). However, again in the concrete example of the Ruijsenaars-Schneider model, the particular form of the occuring matrices enables us to proceed a step further, define a fully algebraic classical YB formulation and eventually connect it with our dynamical quadratic quantum algebras. Let us consider the Lax-matrix structure in the RS A_n case which reads as follows [11, 10]:

$$l = \sum_{i,j} l_{ij} E_{ij} \tag{34}$$

$$l_{ij} = c(q_i - q_j) e^{-p_j} f_j \tag{35}$$

where c and f_j are functions of the position variables. The Poisson-bracket on the phase space is given by $\{p_i, q_j\} = \delta_{ij}$.

The quadratic structure coefficients read [11]:

$$a = -u - s + s^\pi + w - \mathcal{C}, \quad b = -s^\pi - w,$$
$$c = -s + w, \quad d = -u - w - \mathcal{C} \tag{36}$$

For the hyperbolic model the matrices u, s, w take the form:

$$u = -\sum_{i \neq j} \coth(q_i - q_j)\, E_{ij} \otimes E_{ji}\ , \quad s = \sum_{i \neq j} \frac{1}{\sinh(q_i - q_j)}\, E_{ij} \otimes E_{jj}$$
$$w = \sum_{i \neq j} \coth(q_i - q_j)\, E_{ii} \otimes E_{jj} \tag{37}$$

Now using the fact that the a, b, c, d matrices depend only on the position variables and that L_{ij} depends on p as e^{-p_j}, the Poisson-brackets in the last four lines of (33) can be written as L multiplying a certain sum *from the left*:

$$\{M_{12}, l_3\} = l_3 \sum_k E_{kk}^{(3)} \partial_k M_{12} \tag{38}$$

where M stands for any matrix depending only on the position variables. As a result we can rewrite the Jacobi-identity in a purely algebraic form as follows.

$$
\begin{aligned}
\Big([a_{12}, a_{13}] + [a_{12}, a_{23}] + [a_{13}, a_{23}] \Big)\, l_1 l_2 l_3 - \\
- l_1 l_2 l_3 \Big([d_{12}, d_{13}] + [d_{12}, d_{23}] + [d_{13}, d_{23}] + \\
+ \sum_k h_k^{(1)} \partial_k d_{23} - \sum_k h_k^{(2)} \partial_k d_{13} + \sum_k h_k^{(3)} \partial_k d_{12} \Big) \\
+ l_1 l_2 \Big([d_{12}, b_{13}] + [d_{12}, d_{23}] + [b_{13}, b_{23}] + \\
+ \sum_k h_k^{(1)} \partial_k b_{23} - \sum_k h_k^{(2)} \partial_k b_{13} \Big)\, l_3 + \text{circ. perm.} \qquad (39) \\
- l_3 \Big([a_{12}, c_{13}] + [a_{12}, c_{23}] + [c_{13}, c_{23}] - \sum_k h_k^{(3)} \partial_k a_{12} \Big)\, l_1 l_2 - \text{circ.perm.} = 0
\end{aligned}
$$

Hence we may now state:

Theorem 3 *A set of sufficient conditions on the r-matrix which ensures that the Jacobi-identity holds is*

$$
[a_{12}, a_{13}] + [a_{12}, a_{23}] + [a_{13}, a_{23}] = 0 \qquad (40)
$$
$$
[d_{12}, d_{13}] + [d_{12}, d_{23}] + [d_{13}, d_{23}] + \sum_k h_k^{(1)} \partial_k d_{23} - \sum_k h_k^{(2)} \partial_k d_{13} + \qquad (41)
$$
$$
+ \sum_k h_k^{(3)} \partial_k d_{12} = 0
$$
$$
[d_{12}, b_{13}] + [d_{12}, b_{23}] + [b_{13}, b_{23}] + \sum_k h_k^{(1)} \partial_k b_{23} - \sum_k h_k^{(2)} \partial_k b_{13} = 0 \qquad (42)
$$
$$
[a_{12}, c_{13}] + [a_{12}, c_{23}] + [c_{13}, c_{23}] - \sum_k h_k^{(3)} \partial_k a_{12} = 0 \qquad (43)
$$

We are now able to establish a link between these equations and the quantum algebra presented in (5). Indeed, if we assume the existence of a classical limit for A, B, C, D in (13)-(16) as $A = \mathbf{1} + \hbar\, a + \mathrm{O}(\hbar^2)$ ... we can expand the quantum YB-equations (13)-(16) in powers of $\hbar$. The equations (40)-(43) will appear as the first nontrivial term (of order $\hbar^2$) in this expansion.

An example of solution to these classical quadratic YB-equations (40) - (43) is again provided by [5]:

Proposition 1 *The r-matrix of the RS-model defined in (36) verifies these equations.*

This is easily established by direct computations, and does require the additional Casimir terms in a and d. This provides us in addition with a complete algebraic interpretation of the YB-equation for the non-antisymmetric r-matrix of the scalar CM-model obtained as $r = a - c = d - b$ (up to the extra Casimir terms). The full antisymmetric part d and the non-antisymmetric part b must be treated as separate objects obeying (41) and (42). This of course explains why such r-matrices are absent from the classification in [8] where only solutions to (41) are considered.

4   Conclusion

We have proposed a dynamical extension for a general quadratic algebra; we have explicited the consistency conditions as a set of dynamical YB-type equations generalizing the set given in [20], and we have constructed two independent coproduct structures for them.

The next steps are clearly defined: we will look for new explicit quantum solutions of the set (13)-(16) by combining the initial representation (27)-(30) with the coproduct (24) which should lead us to "spin-RS"-like Lax matrices for which this structure would thus provide a suitable algebraic framework; and we will also look for other consistent dynamical extensions of the quadratic algebras. Indeed one already knows at least two such structures: the quantum dynamical Gervais-Neveu-Felder algebras [1, 2] , where $B = C = \mathbf{1}$ and $A = D$ with $h_1 + h_2$ zero-weight condition; and a suggested dynamical version of the reflection algebras, where $A = D^\pi$, $B = C^\pi$ and all objects derive from a single spectral parameter dependent R-matrix [23]. It would be very significant to understand the general scheme, and we hope to report on this soon.

Another issue would be the understanding of (13) - (16) as defining relations for some (quasi-Hopf?) bialgebra , generalizing the construction of [19] for the nondynamical case by suitably incorporating the "coproducts" (24), (26).

Appendix A

We start by writing down the left hand side of equation (12) in its full length.

$$(C_{12'}^{t_{2'}})^{-1}(h_{3'})(D_{1'2'}^{t_{1'}t_{2'}})^{-1}(h_{3'})A_{12}(h_{3'})B_{1'2}^{t_{1'}}(h_{3'}) \times \tag{44}$$
$$(C_{13'}^{t_{3'}})^{-1}(D_{1'3'}^{t_{1'}t_{3'}})^{-1}A_{13}B_{1'3}^{t_{1'}} \times$$
$$(C_{23'}^{t_{3'}})^{-1}(h_{1'})(D_{2'3'}^{t_{2'}t_{3'}})^{-1}(h_{1'})A_{23}(h_{1'})B_{2'3}^{t_{1'}}(h_{1'})$$

We then pick $B_{1'2}^{t_{1'}}(h_{3'})$ and push it through the rest of the product as far as the commutation relations (6) make it possible: in our case $(D_{1'3'}^{t_{1'}t_{3'}})^{-1}$ blocks the way. Now we select the matrix acting on spaces that allow to form a YB-type equation with $B_{1'2}^{t_{1'}}(h_{3'})$ and $(D_{1'3'}^{t_{1'}t_{3'}})^{-1}$. In this case it is $(C_{23'}^{t_{3'}}(h_{1'}))^{-1}$. We indeed can push it through the matrices separating from $(D_{1'3'}^{t_{1'}t_{3'}})^{-1}$ thanks again to the commutation relations (6). We now have to fix a suitable exchange relation for B, D and C. A consistent choice is:

$$B_{1'2}^{t_{1'}}(h_{3'})\ (D_{1'3'}^{t_{1'}t_{3'}})^{-1}\ (C_{23'}^{t_{3'}}(h_{1'}))^{-1} = (C_{23'}^{t_{3'}})^{-1}\ (D_{1'3'}^{t_{1'}t_{3'}})^{-1}\ B_{1'2}^{t_{1'}} \tag{45}$$

which yields after some rearranging, using (7), and total transposition:

$$B_{13}^{t_3}(h_2)\ B_{23}^{t_3}\ D_{12} = D_{12}\ B_{23}^{t_3}(h_1)\ B_{13}^{t_3} \tag{46}$$

Thanks to the commutation relations (6), it is possible to transpose this equation on space 3. The right hand side is to be treated similarly. Eventually we find that requiring (45) amounts to imposing the following dynamical YB-equation on D and B:

$$D_{12}B_{13}B_{23}(\lambda + \gamma h_1) = B_{23}B_{13}(\lambda + \gamma h_2)D_{12} \tag{47}$$

The remaining YB-equations are obtained by repeating the above described process.

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